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# THE INSCRIPTION OF REGULAR POLYGONS.

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## CHAPTER V.

[Continued from the December Number.]

II. *When the number of sides is a multiple of* 5.

*In the regular* 25-*gon*, $A_5-A_{10}=1$, being chords of the regular pentagon. But $A_1-A_2+A_3-A_4+A_5-\ldots\ldots-A_{12}=1$.

$\therefore\quad (A_1-A_4-A_6+A_9+A_{11})+(-A_2+A_3+A_7-A_8-A_{12})=0$.

The product of these two groups expanded is seen to be 0. Hence, each group equals zero.

The sum of the chords $A_1, -A_4, -A_6, A_9, A_{11}$ equals 0; the sum of their 10 products two at a time $=2(A_2-A_3-A_7+A_8+A_{12})+5(A_{10}-A_5)$ $=-5$; the sum of their 10 products three at a time $=-6(A_1-A_4-A_6+A_9+A_{11})+2(A_2-A_3-A_7+A_8+A_{12})=0$; the sum of their 5 products four at a time $=5(A^2-A_5)=5$; the product of all five $=2(A_1-A_4-A_6+A_9+A_{11})-(A_2-A_3-A_7+A_8+A_{12})+A_5=A_5$. Hence, they are the roots of $x^5-5x^3+5x-A_5=0$.

Similarly, $A_2, -A_3, -A_7, A_8, A_{12}$ are the roots of $x^5-5x^3+5x-A_{10}=0$.

*In the regular* 35-*gon*, $A_7-A_{14}=1$; $A_5-A_{10}+A_{15}=1$.

$\therefore\quad (A_2-A_9+A_{12}+A_{16})+(A_6+A_8-A_1-A_{13})+(A_4-A_3-A_{11}-A_{17})=1$.

Write $A$, $B$, $C$ for these three groups respectively.

Then $A+B+C=1$; $AB=3B+3C+4(A_{10}-A_{15})$;

$AC=3A+3B+4(-A_5-A_{15})$; $BC=3A+3C+4(A_{10}-A_5)$.

$\therefore\quad AB+AC+BC=6(A+B+C)+8(-A_5+A_{10}-A_{15})=-2$.

$ABC=C\{3-3A+4(A_{10}-A_{15})\}=3C-3AC+4C(A_{10}-A_{15})$, expanded, $=--1$.

Hence, $A$, $B$, $C$ are the roots of $x^3-x^2-2x+1=0$. But (Chapter I.) $A_5, -A_{10}, A_{15}$ are the roots of this cubic. By inspection, or by a table of natural cosines, we determine which of the roots in the two sets correspond; viz, $A=A_5$; $B=A_{15}$; $C=-A_{10}$.

$$\therefore\quad \begin{cases} A_1-A_6-A_8+A_{13}+A_{15}=0 \\ A_2-A_5-A_9+A_{12}+A_{16}=0 \\ A_3-A_4-A_{10}+A_{11}+A_{17}=0. \end{cases}$$

We may prove by our usual method that:

$A_1, -A_6, -A_8, A_{13}, A_{15}$ are the roots of $x^5-5x^3+5x-A_5=0$

$A_2, -A_5, -A_9, A_{12}, A_{16}$ are the roots of $x^5-5x^3+5x-A_{10}=0$

$A_3, -A_4, -A_{10}, A_{11}, A_{17}$ are the roots of $x^5-5x^3+5x-A_{15}=0$.

By induction, for a regular polygon of $n=5m$ sides:

$$A_1-A_{m-1}-A_{m+1}+A_{2m-1}+A_{2m+1}=0.$$

$$A_2-A_{m-2}-A_{m+2}+A_{2m-2}+A_{2m+2}=0.$$

Generally, $A_s-A_{m-s}-A_{m+s}+A_{2m-s}+A_{2m+s}=0.$

We may prove this trigonometrically by use of the formula for the sum of two cosines; but more elegantly thus:

Since $A_m-A_{2m}=1$, being chords of the regular pentagon,

$$A_s=A_s(A_m-A_{2m})=A_{m-s}+A_{m+s}-A_{2m-s}-A_{2m+s}.$$

The sum of the chords, $A_s, -A_{m-s}, -A_{m+s}, A_{2m-s}, A_{2m+s}$ equals 0; the sum of their products two at a time

$$=2(A_{2s}-A_{m-2s}-A_{m+2s}+A_{2m-2s}+A_{2m+2s})+5(A_{2m}-A_m)=-5;$$

the sum of their products three at a time$=0$; four at a time$=5$; the product of all five$=A_{5s}$. Hence, they are the five roots of $x^5-5x^3+5x-A_{5s}=0$, where $s$ is any integer $\ngtr \frac{m-1}{2}$; and $A_5, -A_{10}, A_{15}, \ldots \pm A_{5s}, \ldots$ are the $\frac{m-1}{2}$ roots of the equation (4) for the regular $m$-gon.

If $m$ is not divisible by 5, one chord of every group of 5 chords is a root of this equation (4). For one and only one of the subscripts $s$, $m-s$, $m+s$, $2m-s$, $2m+s$ is always divisible by 5, as is seen by replacing some by their equivalents: $5m-s$, $m-s$, $4m-s$, $2m-s$, $3m-s$. The remaining four chords will be determined by a series of quadratics whose co-efficients are linear functions of the roots of (4).

Thus, if $s$ be divisible by $5, \pm A_s, \pm A_{2s}, \pm A_{3s}$, are roots of (4).

Then $(-A_{m-s}-A_{m+s})+(A_{2m-s}+A_{2m+s})=-A_s$; $(-A_{m-s}-A_{m+s})(A_{2m-s}+A_{2m+s})=2(-A_m+A_{2m})+(-A_{m-2s}-A_{m+2s}+A_{2m-2s}+A_{2m+2s})=-2-A_{2s}$. $\therefore$ $(-A_{m-s}-A_{m+s})$ and $(A_{2m-s}+A_{2m+s})$ are the roots of the quadratic $x^2+A_sx-(2+A_{2s})=0$.

Now $A_{m-s}\,.\,A_{m+s}=A_{2m}+A_{2s}$; $A_{2m-s}\,.\,A_{2m+s}=A_{2s}-A_m$.

The sum and product of each pair of chords being known in terms of the roots of (4), it follows that, if the $\frac{m-1}{2}$ chords of the regular $m$-gon be found, we can find all the chords of the regular 5 $m$-gon by solving a series of quadratics.

However, if $m$ be divisible by 5, the five chords in any of the above groups are all, or not one of them, roots of equation (4) for the regular $m$-gon; for the subscripts $s$, $m-s$, $m+s$, $2m-s$, $2m+s$, are either all or not one of them divisible by 5, according as $s$ is or is not divisible by 5. Hence, we can not avoid or lower the above quintic.

*The regular* 5-*m gon depends for inscription upon the same equations* *as does the regular m-gon, if m be prime to 5; but also upon one or more quintics of the above form, if m contains the factor 5.*

III. *When the number of sides is divisible by 7.*

For a regular polygon of $n=7m$ sides:

$$A_s-A_{m-s}-A_{m+s}+A_{2m-s}+A_{2m+s}-A_{3m-s}-A_{3m-s}=0.$$

For, $A_m-A_{2m}+A_{3m}=1$, being chords of the regular 7-gon.

Hence, $A_s=A_s(A_m-A_{2m}+A_{3m})=A_{m-s}+A_{m+s}-A_{2m-s}-A_{2m+s}+A_{3m-s}+A_{3m+s}$.

By the usual method of proof, $A_s, -A_{m-s}, -A_{m+s}, A_{2m-s}, A_{2m+s}, -A_{3m-s}, -A_{3m+s}$ are the 7 roots of $x^7-7x^5+14x^3-7x-A_{7s}=0$, where $s \overset{=}{>} \frac{m-2}{2}$, and $A_7, -A_{14}, A_{21}, \ldots \pm A_{7s}, \ldots$ are the roots of the equation (4) for the regular $m$-gon.

If $m$ is prime to 7, one and only one chord of each of the above groups of 7 chords is a root of this equation (4).

The remaining six chords will be determined by a cubic and three quadratics, whose coefficients are linear functions of the roots of (4).

Thus, if $s$ be divisible by 7, $\pm A_s, \pm A_{2s}, \pm A_{3s}, \ldots$ are roots of (4).

Write $A$ for $=A_{m-s}-A_{m+s}$, $B$ for $A_{2m-s}+A_{2m+s}$, $C$ for $-A_{3m-s}-A_{3m+s}$.

Then $A+B+C=-A_s$; $AB+AC+BC=-4(A_m-A_{2m}+A_{3m}) +2(-A_{m-2s}-A_{m+2s}+A_{2m-2s}+A_{2m+2s}-A_{3m-2s}-A_{3m+2s})=-4-2A_{2s}$.

$ABC$ expanded gives $3A_s+A_{3s}$. Hence, $A, B, C$ are the roots of $x^3+A_sx^2-(4+2A_{2s})x-(3A_s+A_{3s})=0$.

But $A_{m-s} \cdot A_{m+s}=A_{2m}+A_{2s}$, etc. Hence, if the $\frac{m-1}{2}$ chords of the regular $m$-gon be found, we can find all the chords of the regular $7m$-gon by solving a cubic and 3 quadratics.

However, if $m$ be divisible by 7, the 7 chords in any of the above groups are all or none of them roots of equation (4) for the regular $m$-gon. Hence, we can neither lower the above septics nor avoid them

*The regular 7m-gon depends for inscription upon the same equation as the regular m-gon, together with an additional cubic, if m be prime to 7; but together with one or more additional septics, if m contains the factor 7.*